On the critical exponent in an isoperimetric inequality for chords

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The problem of maximizing the L^p norms of chords connecting points on a closed curve separated by arclength u arises in electrostatic and quantum–mechanical problems. It is known that among all closed curves of fixed length, the unique maximizing shape is the circle for $1 \le p \le 2$, but this is not the case for sufficiently large values of p. Here we determine the critical value $p_c(u)$ of p above which the circle is not a local maximizer finding, in particular, that $p_c(\frac{1}{2}L) = \frac{5}{2}$. This corrects a claim made in [3].

If $\Gamma(s)$ describes a planar curve of parametrized by arclength s and L is its total length, then

$$\left(\frac{1}{L}\int\limits_0^L |\Gamma(s+u)-\Gamma(s)|^p \mathsf{d}s\right)$$

describes the L^p-mean of the Euclidean length of the chords connecting points separated by arclength u. A reasonable geometric question is to determine the shape that maximizes this quantity for any given value of p. Some phys-

ical phenomena have recently been shown to have connections to this geometric question:

1. What shape will a loop in $\mathbb{R}^3$ carrying a uniform electric charge assume at equilibrium? That is, what is the minimum of the potential energy due to Coulomb repulsion? For this problem see [1, 4] and references therein.

2. What is the shape of a loop Γ of length L that maximizes the ground-state energy of a leaky quantum graph in the plane? That is, how can the fundamental eigenvalue of the leaky-graph Hamiltonian $-\Delta - \alpha\delta(x-\Gamma)$ acting in $L^2(\mathbb{R}^2)$ be maximized? This problem was considered in [2, 3] and references therein.

In both of these problems it turns out that the solution reduces to considering the L^p-means of chords, specifically to establishing the validity of

$$C_L^p(u): \quad c_\Gamma^p(u) := \int_0^L |\Gamma(s+u)-\Gamma(s)|^p \mathsf{d}s \le \frac{L^{1+p}}{\pi^p}\sin^p\frac{\pi u}{L},$$

with $p=1$ and $u \in (0, \frac{1}{2}L]$. In other words, can it be shown that the global maximizer is a planar circle of radius $\frac{1}{2\pi}L$, which by an elementary calculation attains the value on the right side? By a convexity argument it suffices to prove the inequality for any larger value of p to establish it for smaller values.

The inequality $C_L^p(u)$ was established for the first time over forty years ago by Lükő [5] for $p=2$. The same claim was demonstrated more recently in different ways in [1, 3]; see also a local proof in [2]. It is natural to consider the maximal value of p for which the inequality holds. The best upper estimate so far, $p \approx 3.15$, was obtained in [3] by investigating a stadium-shaped Γ.

Our aim here is to improve this result. Using the method of [2] we shall show that among all planar closed curves, $c_\Gamma^p(u)$ is locally maximized by a circle if $p < \frac{5}{2}$, and to find a local critical value of p for "shorter" chords. Since the inequality in question has obvious scaling properties, it is sufficient to consider the case $L = 2\pi$. We keep a general L in the main claims for the convenience of the reader, but otherwise we will work with the particular value $L = 2\pi$.

Without loss of generality we may assume that the Γ is a $\mathcal{C}^2$-smooth curve, the validity of the result being extended to less regular loops by continuity.

Using the notation of [2, Sec. 5] the quantity $c_\Gamma^p(u)$ can be cast into the form

$$c_\Gamma^p(u) = \int\limits_0^L \mathsf{d}s \left[\int\limits_s^{s+u} \mathsf{d}s' \int\limits_s^{s+u} \mathsf{d}s'' \cos\left(\int\limits_{s'}^{s''} \gamma(\tau)\mathsf{d}\tau \right) \right]^{p/2},$$

where $\gamma := \dot{\Gamma}_2\ddot{\Gamma}_1 - \dot{\Gamma}_1\ddot{\Gamma}_2$ is the signed curvature of Γ. Recall that the knowledge of γ allows us reconstruct Γ up to Euclidean transformations by

$$\Gamma(s) = \left(\int_0^s \cos\beta(t)\,\mathsf{d}t, \int_0^s \sin\beta(t)\,\mathsf{d}t \right), \tag{1}$$

where $\beta(s) := \int_0^s \gamma(t)\,\mathsf{d}t$ is the angle between the tangent vectors at $t = s$ and the initial point, $t = 0$. We shall refer to this as the *bending* of the arc.

Our aim is to compute the first and second Gâteaux derivatives of the map $\Gamma \mapsto c_\Gamma^p(u)$ at the circle, $\Gamma = C$, and to demonstrate the claim by looking into their properties. Consequently, we shall consider gentle deformations of a circle, which can be characterized by variations of the curvature

$$\gamma(s) = \frac{2\pi}{L} + \varepsilon g(s), \tag{2}$$

where g is a continuous L-periodic function and ε is small in the sense that $\varepsilon\|g\|_\infty \ll 1$. The periodicity and continuity make it possible to express g through its Fourier series

$$g(s) = a_0 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{2\pi ns}{L}\right) + b_n \cos\left(\frac{2\pi ns}{L}\right)$$

with $\{a\}$, $\{b\} \in \ell^2$. We are interested in closed curves Γ, so we ask now how this property is reflected in Fourier series.

Proposition 1 *The tangent to $\Gamma \in \mathcal{C}^2$ corresponding to (2) is periodic with period L if and only if $a_0 = 0$. Furthermore, $\Gamma(0) = \Gamma(L) + \mathcal{O}(\varepsilon^3)$ provided that*

$$a_1 = b_1 = 0 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{b_n b_{n+1} + a_n a_{n+1}}{n(n+1)} = \sum_{n=2}^{\infty} \frac{a_{n+1} b_n - b_{n+1} a_n}{n(n+1)} = 0.$$

Proof: As mentioned above, we may henceforth set $L = 2\pi$. In view of the definition of $\beta(s)$ it is clear that the tangent vector is continuous if $\beta(L) = 2\pi$. In our case the bending function is

$$\beta(s) = s + \varepsilon \int_0^s g(t)\mathsf{d}t =: s + \varepsilon b(s),$$

and the condition simplifies to $\int_0^{2\pi} g(t)\mathsf{d}t = 0$ which holds *iff* $a_0 = 0$. In view of (1) the fact that Γ is closed means

$$\left(\int_0^{2\pi} \cos\beta(s)\,\mathsf{d}s,\, \int_0^{2\pi} \sin\beta(s)\,\mathsf{d}s \right) = (0,\, 0).$$

For the terms on the right side of the last equation we have the expansion

$$\begin{aligned} \cos\beta(s) &= \left(1 - \frac{1}{2}\varepsilon^2 b^2(s)\right) \cos s - \varepsilon b(s) \sin s + \mathcal{O}(\varepsilon^3), \\ \sin\beta(s) &= \left(1 - \frac{1}{2}\varepsilon^2 b^2(s)\right) \sin s + \varepsilon b(s) \cos s + \mathcal{O}(\varepsilon^3). \end{aligned}$$

Up to the third order in ε we get thus the conditions

$$\int_0^{2\pi} b(s)\, \cos s\, \mathsf{d}s = \int_0^{2\pi} b(s)\, \sin s\, \mathsf{d}s = 0, \tag{3}$$

$$\int_0^{2\pi} b(s)^2\, \cos s\, \mathsf{d}s = \int_0^{2\pi} b(s)^2\, \sin s\, \mathsf{d}s = 0. \tag{4}$$

It is convenient to rewrite the Fourier series for the curvature deformation in the complex form, $g(s) = \sum_{n\neq 0} c_n\, e^{ins}$ where $c_{-n} = \bar{c}_n$ and for $n > 0$ we have $c_n = \frac{1}{2}(b_n - ia_n)$. For $b(s)$ this yields the following series:

$$b(s) = \sum_{n\neq 0} \frac{ic_n}{n} \left[1 - e^{ins}\right].$$

Using orthonormality of the trigonometric basis we see that the condition (3) requires $a_1 = b_1 = 0$. On the other hand, the remaining condition (4) means that the integral $\int_0^L b(s)^2\, e^{ins}\, \mathsf{d}s$ must vanish; with the help of the above series we can express it in the following way,

$$-\sum_{n,\, m\neq 0,\pm 1} \frac{c_n c_m}{nm} \int_0^{2\pi} e^{is} \left[1 - e^{ins}\right] \left[1 - e^{ims}\right] \mathsf{d}s = \sum_{n\neq 0,\pm 1} \frac{c_n \bar{c}_{n+1}}{n(n+1)}\,,$$

and taking the real and imaginary part we arrive at the claimed identities for $\{a_n\}$ and $\{b_n\}$. ■

After this preliminary let us turn to our proper subject. The Gâteaux derivative of the functional (1) in the direction g is

$$\mathsf{D}_g c_\Gamma^p(u) = \left.\frac{\partial c_\Gamma^p(u)}{\partial\varepsilon}\right|_{\varepsilon=0}$$
$$= -\frac{p}{2}\left[4\sin^2\frac{u}{2}\right]^{p/2-1}\int\limits_0^{2\pi}\mathsf{d}s\int\limits_s^{s+u}\mathsf{d}s'\int\limits_s^{s+u}\mathsf{d}s''\sin\left(\int\limits_{s'}^{s''}\mathsf{d}t\right)\int\limits_{s'}^{s''}g(\tau)\,\mathsf{d}\tau \quad (5)$$

again for $L = 2\pi$, and the second derivative is

$$\mathsf{D}_g^2 c_\Gamma^p(u) = \left.\frac{\partial^2 c_\Gamma^p(u)}{\partial\varepsilon^2}\right|_{\varepsilon=0}$$
$$= \frac{p}{2}\left(\frac{p}{2}-1\right)\left[4\sin^2\frac{u}{2}\right]^{p/2-2}\int\limits_0^{2\pi}\mathsf{d}s\left(\int\limits_s^{s+u}\mathsf{d}s'\int\limits_s^{s+u}\mathsf{d}s''\sin(s''-s')\int\limits_{s'}^{s''}g(\tau)\,\mathsf{d}\tau\right)^2$$
$$-\frac{p}{2}\left[4\sin^2\frac{u}{2}\right]^{p/2-1}\int\limits_0^{2\pi}\mathsf{d}s\int\limits_s^{s+u}\mathsf{d}s'\int\limits_s^{s+u}\mathsf{d}s''\cos(s''-s')\left(\int\limits_{s'}^{s''}g(\tau)\,\mathsf{d}\tau\right)^2. \quad (6)$$

Rearranging the integrals in (5) we get

$$\int\limits_0^{2\pi}\mathsf{d}s\int\limits_s^{s+u}\mathsf{d}s'\int\limits_s^{s+u}\mathsf{d}s''\sin(s''-s')\int\limits_{s'}^{s''}g(\tau)\,\mathsf{d}\tau$$
$$= \int\limits_0^{2\pi}\mathsf{d}\tau\int\limits_{\tau-u}^{\tau}\mathsf{d}s\int\limits_s^{\tau}\mathsf{d}s'\int\limits_\tau^{s+u}\mathsf{d}s''\sin(s''-s')\,g(\tau)\,\mathsf{d}\tau$$
$$= \left(4\sin^2 u + u\sin u\right)\int\limits_0^L g(\tau)\,\mathsf{d}\tau = 0,$$

which shows that for every $p > 0$ the circle is either an extremal or a saddle point. (There are no solutions to $4\sin u = -u$ in $[-\pi,\pi]$.) In the next step

we analyze the second derivative to distinguish in between these two cases. Not surprisingly, the answer depends on the value of u. Our main result reads

Theorem 2 *For a fixed arc length $u \in (0, \frac{1}{2}L]$ define*

$$p_c(u) := \frac{4 - \cos\left(\frac{2\pi u}{L}\right)}{1 - \cos\left(\frac{2\pi u}{L}\right)}, \tag{7}$$

then we have the following alternative. For $p > p_c(u)$ the circle is either a saddle point or a local minimum, while for $p < p_c(u)$ it is a local maximum of the map $\Gamma \mapsto c_\Gamma^p(u)$.

Before passing to the proof let us make a pair of comments.

Remarks 3

1. It will be seen from the proof that in the critical case $p = p_c(u)$, the higher order derivatives of $c_\Gamma^p(u)$ come into play. We shall not address the critical case here.

2. It is natural to expect and easy to verify that for $p > p_c$ circle is in fact a saddle point of the functional.

Proof: We put again $L = 2\pi$ and analyze the terms of the second derivative (6) separately. By a straightforward computation using orthonormality of the trigonometric basis the iterated integral in the first term equals

$$\sum_{n=2}^{\infty} \left[a_n^2 \mathsf{fs}_1(n,\, u,\, p) + b_n^2 \mathsf{fc}_1(n,\, u,\, p)\right],$$

where

$$\mathsf{fs}_1(n,\, u,\, p) = \mathsf{fc}_1(n,\, u,\, p) := \frac{16\pi}{n - n^3}\left(-2n \cos\frac{nu}{2} \sin^2\frac{u}{2} + \sin u \sin\frac{nu}{2}\right)^2.$$

In the second term we rearrange the integrals before using the Fourier series,

$$\begin{aligned}
&\int\limits_{s}^{s+u} \mathsf{d}s' \int\limits_{s}^{s+u} \mathsf{d}s'' \int\limits_{s'}^{s''} \mathsf{d}\tau \int\limits_{s'}^{s''} \mathsf{d}\tau' \cos(s''-s')\, g(\tau)g(\tau') \\
&\qquad = 2 \int\limits_{s}^{s+u} \mathsf{d}\tau \int\limits_{\tau}^{s+u} \mathsf{d}\tau' \int\limits_{s}^{\tau} \mathsf{d}s' \int\limits_{\tau'}^{s+u} \mathsf{d}s'' \cos(s''-s')\, g(\tau)g(\tau') \\
&\qquad\qquad =: \int\limits_{s}^{s+u} \mathsf{d}\tau \int\limits_{\tau}^{s+u} \mathsf{d}\tau'\, g(\tau)g(\tau')\, \mathsf{Int}(s,\,\tau,\,\tau').
\end{aligned}$$

Hence the full integral in the second term of (6) equals

$$\begin{aligned}
&\int\limits_{0}^{2\pi} \mathsf{d}s \int\limits_{s}^{s+u} \mathsf{d}\tau \int\limits_{\tau}^{s+u} \mathsf{d}\tau'\, g(\tau)g(\tau')\, \mathsf{Int}(s,\,\tau,\,\tau') \\
&= \int\limits_{0}^{2\pi} \mathsf{d}\tau \int\limits_{\tau}^{\tau+u} \mathsf{d}\tau' \int\limits_{\tau'-u}^{\tau} \mathsf{d}s\, g(\tau)g(\tau')\, \mathsf{Int}(s,\,\tau,\,\tau') =: \int\limits_{0}^{2\pi} \mathsf{d}\tau \int\limits_{\tau}^{\tau+u} \mathsf{d}\tau'\, \mathsf{Int}_2(\tau,\,\tau')\, g(\tau)g(\tau'),
\end{aligned}$$

where

$$\mathsf{Int}_2(\tau,\,\tau') := 2(\tau'-\tau-u)\big(\cos(\tau'-\tau)+\cos u\big) + 4\big(-\sin(\tau'-\tau)+\sin u\big).$$

Finally we use the Fourier series and obtain an expression for the iterated integral in the second term

$$\int\limits_{0}^{2\pi} \mathsf{d}s \int\limits_{s}^{s+u} \mathsf{d}s' \int\limits_{s}^{s+u} \mathsf{d}s'' \cos(s''-s') \left(\int\limits_{s'}^{s''} g(\tau)\mathsf{d}\tau \right)^2 = \sum_{n=2}^{\infty} \left[a_n^2 \mathsf{fs}_2(n,\,u,\,p) + b_n^2 \mathsf{fc}_2(n,\,u,\,p) \right],$$

where

$$\begin{aligned}
\mathsf{fs}_2(n,\,u,\,p) = \mathsf{fc}_2(n,\,u,\,p) := \frac{\pi}{n-n^3} \big(&-6n^2+2n^4-2(n^2-1)^2\cos u \\
&+(n+1)^2\cos(n-1)u + (n-1)^2\cos(n+1)u\big).
\end{aligned}$$

Now we put it together and get the second derivative in the form

$$\mathsf{D}_g^2 c_\Gamma^p(u) = \sum_{n=2}^{\infty} (a_n^2 + b_n^2)\, \frac{2^p \pi \sin^{p-2}\left(\frac{u}{2}\right)}{8(n-n^3)^2}\, p\, T(n,\, u,\, p), \tag{8}$$

where

$$T(n,\, u,\, p) := -\Big(2n^4 - 6n^2 - 2(n^2-1)^2 \cos u + (n+1)^2 \cos(n-1)u$$
$$+(n-1)^2 \cos(n+1)u\Big) + 2(p-2)\left(-2n \cos\left(\frac{nu}{2}\right) \sin\left(\frac{u}{2}\right) + 2\cos\left(\frac{u}{2}\right) \sin\left(\frac{nu}{2}\right)\right)^2. \tag{9}$$

Since $\sin(u/2)$ is positive for $u \in (0,\, \pi)$, the sign of each term in the second derivative series (8) is determined by that of $T(n,\, u,\, p)$. The equation

$$T(2,\, u,\, p) = -16\big(4 - p + (p-1)\cos u\big) \sin^4\left(\frac{u}{2}\right)$$

gives $T(2,\, u,\, p) > 0$ for $p > p_c(u)$, proving the easier part of the alternative, namely that for $p > p_c(u)$ the circle fails to be a local maximum of the map $\Gamma \mapsto c_\Gamma^p(u)$.

It is easy to check that $T(n,\, u,\, p)$ is strictly increasing as a function of p. Hence to prove the other part of the theorem it is sufficient to show that $T(n,\, u,\, p_c(u))$ is negative for $n \geq 3$. To this aim we define

$$S(n,\, u) = -(1 - \cos u)\, T(n,\, u,\, p_c(u))\,;$$

we next prove that this function is positive for $n \geq 3$.

Inserting the critical exponent $p_c(u)$ into (9) we obtain

$$\begin{aligned} S(n,\, u) = -4 - 10n^2 + 2n^4 + 2(n^2-1)\big(-2(n^2-2)\cos u + n^2 \cos^2 u\big) \\ + 4\cos(nu)\big(1 - n^2 + (2+n^2)\cos u\big) + 12n \sin u \sin(nu), \end{aligned}$$

and using the inequality $(a \sin x + b \cos x)^2 \leq a^2 + b^2$ we get the bound

$$\begin{aligned} S(n,\, u) \geq -4 - 10n^2 + 2n^4 + 2(n^2-1)\big(-2(n^2-2)\cos u + n^2 \cos^2 u\big) \\ - 4\sqrt{\big(1 - n^2 + (2+n^2)\cos u\big)^2 + 9n^2 \sin^2 u}. \end{aligned}$$

Hence $S(n,\, u)$ is positive whenever

$$-4 - 10n^2 + 2n^4 + 2(n^2-1)\big(-2(n^2-2)\cos u + n^2 \cos^2 u\big) > 0 \tag{10}$$

and

$$\Big(-4-10n^2+2n^4+2(n^2-1)\big(-2(n^2-2)\cos u+n^2\cos^2 u\big)\Big)^2$$
$$> 16\Big(\big(1-n^2+(2+n^2)\cos u\big)^2+9n^2\sin^2 u\Big). \quad (11)$$

The first condition (10) is a quadratic equation in $\cos u$, and a calculation shows that it is satisfied for $\cos u < 1-\frac{6}{n^2}$. Using the notation $\cos u = x$, the second condition (11) simplifies to

$$4n^2(n^2-1)^2(8+n^2(x-1))(x-1)^3 > 0,$$

which provides us with a slightly stronger condition,

$$\cos u < 1-\frac{8}{n^2}\,. \quad (12)$$

The vicinity of zero has to be regarded separately to prove the positivity of $S(n,\,u)$ on the interval complementary to (12) . By a straightforward computation the Taylor expansion of $S(n,\,u)$ around zero equals

$$S(n,\,u)=\frac{n^2u^8}{40}\left(-\frac{1}{9}+\frac{n^2}{4}-\frac{n^4}{6}+\frac{n^6}{36}\right)+\frac{u^{10}}{10!}R_{10}, \quad (13)$$

where for $n\geq 3$ and u in the complement of (12) the $\mathcal{O}(u^{10})$ term is bounded from below by

$$R_{10}\geq -136n^{10}\,.$$

Comparing the reminder with the first term on the right-side of (13), we observe that $S(n,\,u)$ is positive for

$$u^2<\frac{1}{40}\left(-\frac{1}{9}+\frac{n^2}{4}-\frac{n^4}{6}+\frac{n^6}{36}\right)\frac{10!}{136n^8}\,. \quad (14)$$

Now we use the inequality $\cos u\leq 1-7/16u^2$ for $u\in(0,\frac{6}{5})$ to compare the intervals (12) and (14). By simple analysis we find out that for $n\geq 4$,

$$1-\frac{8}{n^2}\leq 1-\frac{7}{16}\frac{1}{40}\left(-\frac{1}{9}+\frac{n^2}{4}-\frac{n^4}{6}+\frac{n^6}{36}\right)\frac{10!}{136n^8}\,,$$

and hence in this case the union of the intervals covers $(0,\,\pi)$, which proves that $S(n,\,u)\geq 0$ holds for $n\geq 4$.

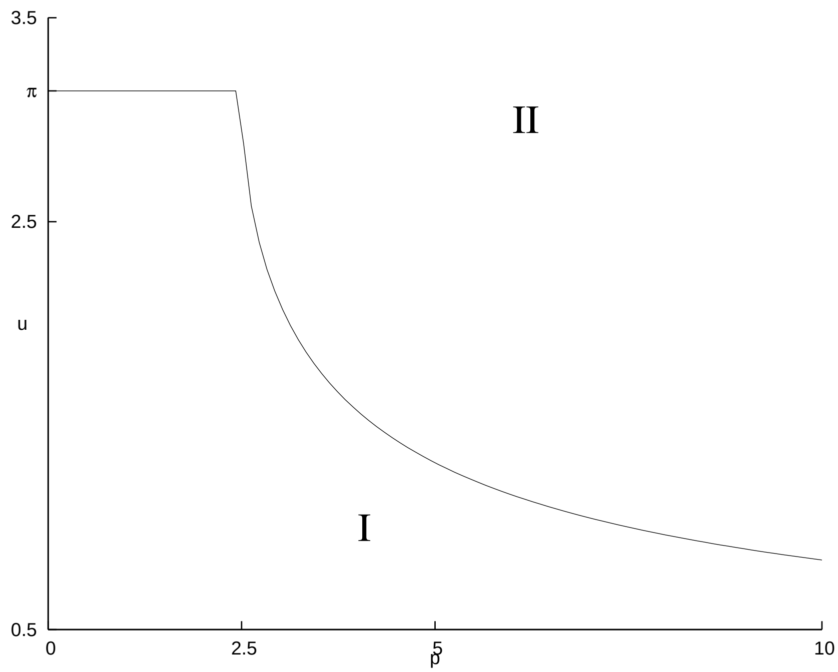

Figure 1: The relation between the critical exponent p_c and the arc length u. The mean-chord inequalities hold locally in the region I.

In the case $n = 3$ the positivity of $S(n, u)$ is easily established, as the function $S(3, u)$ simplifies now to

$$S(3, u) = 2\left(2\sin\frac{u}{2}\right)^8 .$$

Since $T(2, u, p) < 0$ holds for $p < p_c(u)$ the theorem is proven. ■

To visualize the result, in Figure 1 we plot the relation between the critical exponent p_c given by (7) and the arc length u.

A comment is due on the closure of the curve Γ. In [2] the local validity of the inequality for $p = 2$ was proved without this hypothesis. Here we used closure, but not to the full power of Proposition 1. We relied simply on the fact that the Fourier coefficients vanish for $|n| \leq 1$, which meant that the endpoints $\Gamma(0)$ and $\Gamma(2\pi)$ meet within an error of $\mathcal{O}(\varepsilon^2)$, not $\mathcal{O}(\varepsilon^3)$.

Let us finally make one more remark, namely on a claim made in Thm. 5.4

of [3]. It was stated there that for a particular class of deformations the circle remains a local maximizer for all p, namely for those which, in the complex notation, have the form $(1-\varepsilon)e^{is}+\Theta(\varepsilon,\, s)$, with the assumption that for each ε, $\Theta(\varepsilon,\, s)$ is orthogonal to e^{is} and $\Theta(\varepsilon,\, s)$ is $\mathcal{C}^2$ smooth. In fact, the $\mathcal{C}^2$ assumption in the variable ε cannot occur. To see that, notice that the condition $\int |\dot\Gamma(s)|^2\, \mathsf{d}s = 2\pi$ together with orthogonality imply

$$\int |\Theta_s(\varepsilon,\, s)|^2\, \mathsf{d}s = 4\pi\varepsilon - 2\pi\varepsilon^2,$$

where $\Theta_s := \partial\Theta/\partial s$. Since Θ is $\mathcal{C}^2$ by assumption, we may differentiate under the integral sign to get

$$2\mathrm{Re}\int \bar\Theta_s(\varepsilon,\, s)\frac{\partial\Theta_s(\varepsilon,\, s)}{\partial\varepsilon}\, \mathsf{d}s = 4\pi - 4\pi\varepsilon\,;$$

using the observation from [3] that $\Theta(0,\, s) = 0$ we see that the left-hand side would tent to zero as $\varepsilon \to 0$ given the assumption that Θ is jointly $\mathcal{C}^2$, while the right-hand one has the nonzero limit 4π. To obtain smooth perturbations one should suppose, e.g., $\Gamma(\varepsilon,\, s) = (1-\varepsilon^2)e^{is}+\Theta(\varepsilon,\, s)$, and this would necessitate an analysis to second order in ε, as has been done in this article.

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